NOTE

BIER SPHERES AND BARYCENTRIC SUBDIVISION

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With any proper simplicial subcomplex K of the abstract $(n-1)$-simplex $2^{\{1,\dots,n\}}$, one can associate the *Bier sphere* $\mathrm{Bier}_n(K)$, a certain triangulated (piecewise linear) $(n-2)$-sphere [1]. The Bier spheres have become an established tool in topological combinatorics and received attention in connection with the van Kampen–Flores Theorem and the existence of non-polytopal triangulations of spheres [2, 3, 4, 5, 6].

It is not obvious that Bier spheres are actually spheres. The proof in [6, pp. 112–116] is a recursive construction using the bistellar flip operation on triangulations. This note provides a short non-recursive proof bringing to light some of the inherent structure of the Bier spheres.

As a reference to all mentioned topological concepts we refer once again to the wonderful book [6] by Jiří Matoušek.

The combinatorial Alexander dual. Let $K \subset 2^{[n]}$ be an abstract simplicial complex. The *combinatorial Alexander dual* $K^\star$ *of* K is given by the complements of the non-faces of K:

$$K^\star = \{\tau \subset [n] : \overline{\tau} \notin K\},$$

where $\overline{\tau}$ denotes the complement $[n] \setminus \tau$. The following lemma characterizes the Alexander dual via its first barycentric subdivision as illustrated in Figure 1. Let's denote the first barycentric subdivision of a complex L by $\mathrm{sd}(L)$. Recall that the simplices of $\mathrm{sd}(L)$ are given by (possibly empty) inclusion chains of nonempty simplices of L.

Lemma 1. For any simplicial complex $K \subset 2^{[n]}$ we have

$$\{t_0 \subset \cdots \subset t_l\} \in \mathrm{sd}(K^\star) \subseteq \mathrm{sd}(2^{[n]} \setminus \{[n]\})$$

if and only if $\{\overline{t_l} \subset \cdots \subset \overline{t_0}\}$ is disjoint from $\mathrm{sd}(K) \subseteq \mathrm{sd}(2^{[n]} \setminus \{[n]\})$. $\square$

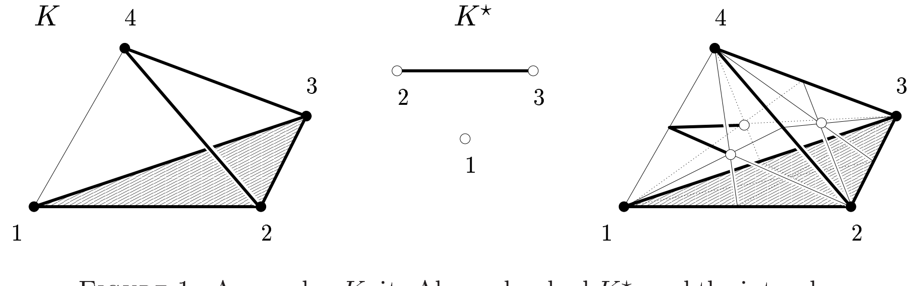

FIGURE 1. A complex K, its Alexander dual $K^\star$, and the interplay of their barycentric subdivisions

Subcomplexes of joins. In the following we denote the join operation as usual by $*$.

Lemma 2. Let $K \subseteq 2^{[n]}$ and $L \subseteq 2^{[m]}$ be two abstract simplicial complexes, and J be a subcomplex of the join $K * L$. Then the complexes J and

$$\{s * t : s \in \mathrm{sd}(2^\sigma), t \in \mathrm{sd}(2^\tau), \sigma * \tau \in J\} \subseteq \mathrm{sd}(K) * \mathrm{sd}(L)$$

possess identical geometric realizations. □

The Bier sphere. For $K \subset 2^{[n]}$ the Bier sphere $\mathrm{Bier}_n(K)$ is defined as the deleted join of K with its combinatorial Alexander dual

$$\begin{aligned}\mathrm{Bier}_n(K) = K *_\Delta K^\star &= \{\sigma * \tau : \sigma \in K, \tau \in K^\star, \sigma \cap \tau = \emptyset\} \\ &= \{\sigma * \tau : \sigma \in K, \overline{\tau} \notin K, \sigma \subseteq \overline{\tau}\}.\end{aligned}$$

Lemma 1 and Figure 1 suggest the following.

Theorem. *For a simplicial complex $K \subset 2^{[n]}$ the Bier sphere $\mathrm{Bier}_n(K)$ is a (piecewise linear) $(n-2)$-sphere. More precisely, if we replace each simplex $\sigma * \tau$ in the Bier sphere by $\mathrm{sd}(2^\sigma) * \mathrm{sd}(2^\tau)$ we obtain a complex isomorphic to $\mathrm{sd}(2^{[n]} \setminus \{[n]\})$, i.e., the barycentric subdivision of the boundary of the $(n-1)$-simplex.*

Proof. According to Lemma 2 $\mathrm{Bier}_n(K) = K *_\Delta K^\star$ and the following complex possess identical geometric realizations.

$$\begin{aligned}&\{s * t : s \in \mathrm{sd}(2^\sigma), t \in \mathrm{sd}(2^\tau), \sigma * \tau \in K *_\Delta K^\star\} \\ &\quad = \{(s_1 \subset \cdots \subset s_k) * (t_1 \subset \cdots \subset t_l) : \emptyset \neq s_i \subseteq \sigma, \emptyset \neq t_j \subseteq \tau, \sigma \in K, \overline{\tau} \notin K, \sigma \subseteq \overline{\tau}\}\end{aligned}$$

The last complex maps isomorphically to $\mathrm{sd}(2^{[n]} \setminus \{[n]\})$ via

$$(s_1 \subset \cdots \subset s_k) * (t_1 \subset \cdots \subset t_l) \mapsto (s_1 \subset \cdots \subset s_k \subset \overline{t_l} \subset \cdots \subset \overline{t_1}).$$

This map is well defined and injective since $s_k \subseteq \sigma \subseteq \overline{\tau} \subseteq \overline{t_l} \subset [n]$. To see surjectivity let $u_1 \subset \cdots \subset u_m \in \mathrm{sd}(2^{[n]} \setminus \{[n]\})$. Let $u_0 = \emptyset$, $u_{m+1} = [n]$, and $k \geq 0$ be maximal with the property that $u_k \in K$. Then define σ to be u_k and τ to be $\overline{u_{k+1}}$. Set $s_1 = u_1, \ldots, s_k = u_k$ and $t_1 = \overline{u_m}, \ldots, t_{m-k} = \overline{u_{k+1}}$. Then trivially

- ▷ $s_1 \subset \cdots \subset s_k \subseteq \sigma$,
- ▷ $t_1 \subset \cdots \subset t_{m-k} \subseteq \tau$,
- ▷ $\sigma \in K, \overline{\tau} \notin K, \sigma \subset \overline{\tau}$, and
- ▷ $(s_1 \subset \cdots \subset s_k) * (t_1 \subset \cdots \subset t_{m-k}) \mapsto u_1 \subset \cdots \subset u_m$,

which finishes the proof. □

References

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